

On a Microscopic Representation of Space-Time V

Rolf Dahm¹

¹*Permanent address: beratung für Informationssysteme und Systemintegration,
Gärtnergasse 1, D-55116 Mainz, Germany*

In order to extend our approach based on $SU^*(4)$, we were led to (real) projective and (line) Complex geometry. So here we start from quadratic Complexes which yield naturally the 'light cone' $x_1^2 + x_2^2 + x_3^2 - x_0^2 = 0$ when being related to (homogeneous) point coordinates x_α^2 and infinitesimal dynamics by tetrahedral Complexes (or line elements). This introduces naturally projective transformations by preserving anharmonic ratios. Referring to old work of Plücker relating quadratic Complexes to optics, we discuss (linear) symplectic symmetry and line coordinates, the main purpose and thread within this paper, however, is the identification and discussion of special relativity as direct invariance properties of line/Complex coordinates as well as their relation to 'quantum field theory' by complexification of point coordinates or Complexes. This can be established by the Lie mapping which relates lines/Complexes to sphere geometry so that $SU(2)$, $SU(2) \times U(1)$, $SU(2) \times SU(2)$ and the Dirac spinor description emerge without additional assumptions. We give a short outlook in that quadratic Complexes are related to dynamics e.g. power expressions in terms of six-vector products of Complexes, and action principles may be applied. (Quadratic) products like $F^{\mu\nu} F_{\mu\nu}$ or $F^{a\mu\nu} F_{\mu\nu}^a$, $1 \leq a \leq 3$ are natural quadratic Complex expressions ('invariants') which may be extended by line constraints $\lambda k \cdot \epsilon = 0$ with respect to an 'action principle' so that we identify 'quantum field theory' with projective or line/Complex geometry having applied the Lie mapping.

PACS numbers: 02.20.-a, 02.40.-k, 03.70.+k, 04.20.-q, 04.50.-h, 04.62.+v, 11.10.-z, 11.15.-q, 11.30.-j, 12.10.-g

I. INTRODUCTION

With this fifth part¹ of our series, we've reached a certain milestone to pause and make up the balance in departing from $SU(4)$ vs. $SU^*(4)$, the various derived symmetry breaking patterns [2], [3] and the identification of photons in SSB patterns. So starting from the (physical!) interpretation of the Lie algebra $\mathfrak{su}(4)$ (in terms of spin, isospin and chiral symmetry) and its various real forms, the main path – also in the context of noncompact $\mathfrak{su}^*(4)$ and the Dirac algebra – concentrated on a $10 \oplus 5$ reductive decomposition and its interpretation (see [3] and references). In collecting various algebraic and physical aspects, we were led to projective geometry, transfer principles and especially line and Complex geometry [4], [5].

Here, we want to present briefly some additional details and identifications with respect to typical/basic 'ingredients' of dynamical theories like special relativity, Dirac's 'square root' of the Klein-Gordon equation, and – most important – identification of states resp. interpretation of the mathematical description. Whereas a 'light cone', Lorentz transformations and special relativity can be related to (real) projective geometry, we need the Lie mapping [14] to introduce Pauli matrices (respectively quaternions) which relates to 'quantum theory' and 'quantum' notion.

The central theme is² that this transformation relates *line* representations³ (NOT *point* reps!) to spheres (and elliptic geometry). So first of all we have to use and respect a priori *two* distinct spaces (and not one and the same space-time) respectively their related individual 'physical' identification(s) and, more important, our fundamental geometrical space element is a (nonlocal) 4-dim *line* and afterwards a set of lines, the linear Complex, not as usual the point or 'vector' rep. So points emerge only as line incidences which must be treated by projective geometry. However, dynamics being concerned with tangents, momenta and conic sections benefits enormously according to this different choice of the fundamental geometrical space element due to (linear) Complexes being related to null systems, the Hamilton formalism and differential equations [19], [15]. This comprises a very long history where lines, various compositions of lines (i.e. Complexes, congruences, etc.) and null systems have been applied with great use and success to physics. So here, we start right from the beginning with line and Complex geometry⁴.

¹ This paper, being limited as conference proceedings, yields mainly the direct arguments and calculations related to the abstract. A more detailed paper containing additional background and details is going to appear as part VI soon.

² Here, we discuss 'simple' geometrical issues and identifications according to [14] and postpone the viewpoint of 'advanced geometry' in terms of the Plücker-Klein quadric in \mathbb{R}^5 , Study's points of view [23], or even higher dimensional transfers to upcoming publications/discussions.

³ As before, subsequently we use the shorthand notation 'rep'.

⁴ For introductory details from physical identifications to projective geometry, please see [4] and [5], some basic

II. QUADRATIC COMPLEXE AND THE 'LIGHT CONE'

Why do we want to see the 'light cone' $x_1^2 + x_2^2 + x_3^2 - x_0^2 = 0$ (in point coordinates) emerging directly from quadratic Complexe like in [4]? Well, the reason is simple: As soon as we interpret this 'light cone' (as usual) in terms of a 'metric' on point spaces and/or in metric coordinates, we have to introduce additional physical identification or at least an additional dimension (i.e. we have to use *five* homogeneous coordinates, see e.g. [13], appendix §5) in order to treat absolute elements ('infinities'). From our viewpoint, it is much easier and much more consistent (and we'll give some additional arguments below) to understand the 'light cone' as an absolute element (or 'gauge surface') when switching from quadratic line/Complex reps to point reps. This allows to introduce and relate velocities via a 'common' (system) time parameter t to point coordinates x_α , i.e. the geometrical line picture to a dynamic point picture and as such to relations/ratios of velocities in terms of $\beta_i = v_i/c$. So the velocity of light c may serve to parametrize the absolute coordinate x_0 by its correspondence to physical observations and identifications, using the velocity picture $x_i \sim v_i t$, of course (see [4]), and the 'gauge surface' in the geometrical line picture to apply the Cayley-Klein mechanism towards metric coordinates and spaces. We have mentioned already the introduction of equivalence classes by this velocity concept. Although it is known from classical mechanics how to work with (euclidean) coordinate projections and 'vectors', it is also known⁵ that in order to describe kinematics and dynamics, the naive (3-dim euclidean) 'vector' picture of velocities, momenta and forces is not sufficient but one has to work with force and null systems which were generalized by Plücker to Complexe and 6-dim 'Dynamen' [18]. This means, we use *four homogeneous* coordinates x_α , and the 'metric'/the 'light cone' is a point representation related to the line representation of a quadratic Complex. So this picture yields a natural relation in the context of projective geometry of 3-space, and natural representations of lines in terms of points and planes (see [4], p. 9, and references).

Here, we skip some background on quadratic Complexe and postpone the details to part VI. In brief, having mentioned ([5], III C) Plücker's work on relating ellipsoids and lines/Complexe already, some more aspects can be found in Klein's book⁶ [12] §§4-7 with respect to potential theory, confocal surfaces and elliptic coordinates. Physical applications in optics [17], [19] should

definitions have been repeated in [4] section 1.2ff which originate mainly from Plücker's work [19] on line Complexe.

⁵ As an example, we cite [21] which, starting from Complex description of rigid bodies, gives insight into some deeper aspects of 'advanced geometry' by addressing null systems and the tetrahedral Complex, 3rd and higher order elements, etc.

⁶ For the sake of simplicity, we use this book and the references given there as a kind of dictionary in order to avoid lengthy details or historical background.

be mentioned, too, in that double refraction at a crystal surface yields a triple (line) vertex or in generalizing the variational action principle [18], [11].

In addition, we have learned from [7] a basic set of requirements to express relativity, and we have to state that – being in charge to introduce those requirements in point reps and differential geometry – it is Complex geometry (and especially in conjunction with quadratic Complexes) which yields the requirements with respect to necessary conformal, projective and affine structures automatically, especially such non-local requirements like second-order cones at each point of the curve or geodesic families with certain behavior. So using quadratic Complexes, we control a superset *to derive* those features – there is no need to introduce them by hand, but we have to use Complex geometry. So the unifying space element is a linear Complex, and we have to relate our reasoning in 3-dim space to higher order Complexes and calculation patterns in order to compare to physics and extract principles.

A. Infinitesimal Dynamics

Recalling tetrahedral Complexes, those quadratic Complexes (while preserving by definition anharmonic ratios, see [5] and references there) emphasize projective transformation, i.e. transformations leaving the anharmonic ratio invariant. This provides the most 'natural link' via Klein's Erlanger program to introduce and justify group theory in terms of linear reps. So departing here, we may study projective transformations in various reps e.g. only on appropriate point sets, their transformation groups or even only infinitesimal transformations related to them⁷. With respect to more general geometric considerations, however, by changing the underlying space element (e.g. from points to lines or spheres), we want to make first steps into linear and quadratic Complex geometry⁸. Additional details can be found in [12] §§21-24.

Now the shorthand approach to dynamics can be based on line incidence and in a second step on Complex involutions [11]. Whereas the incidence relation of two lines with coordinates $p_{\alpha\beta}$ and $p'_{\alpha\beta}$ reads as ([12] §20)

$$p_{12}p'_{34} + p_{13}p'_{42} + p_{14}p'_{23} + p_{34}p'_{12} + p_{42}p'_{13} + p_{23}p'_{14} = 0, \quad (1)$$

⁷ We want to remember that projective geometry of \mathbb{R}^3 not only involves 'collineations' but also 'correlations' and dual elements, and it allows for application of higher transfer principles! In the context of transfer principles, it is necessary to recall the identification of certain *point* coordinates in higher dimensional spaces with (even extended) geometrical elements in ordinary real 3-dim space, e.g. points with respect to the Plücker-Klein quadric or the Lie mapping later in section IV.

⁸ Of course, we may choose other elements and transfer them to (higher-dim) point spaces. Here, we stay in the regime of 3-dim space, the Complex as basic element, the Lie mapping, sphere and Laguerre geometry.

choosing the linear Complex as basic element introduces new notations and invariant structures [12] §§21-24. The incidence equation (1) above, with line coordinates $p_{\alpha\beta}$ being replaced by Complex coordinates, declare involution of Complexes, and in analyzing the quadratic form Ω , Klein introduced [10], [11] six fundamental Complexes and determined their invariants to be ± 1 . Because of their relation to handedness, the six fundamental Complexes⁹ decompose into $3 \oplus 3$ left-/right-handed, the simultaneous invariant of each pair being 0.

If – as before [4] – we identify $F^{\mu\nu}F_{\mu\nu}$ as a Complex invariant, on the one hand we can use Klein’s work to postulate an action (see section IV B), on the other hand one has to recall (see e.g. [16], [18], [10] or [11]) the discussion of up to six acting forces. So with respect to $F^{\mu\nu}F_{\mu\nu}$ and $F^{a\mu\nu}F_{\mu\nu}^a$, $1 \leq a \leq 3$, emerging in actions, we have settled the basic environment here from within the framework of Complex geometry and involutions.

Last not least, it is noteworthy that the (infinitesimal) action (and as such the very foundation of the action principle) may be written as a quadratic Complex (see e.g. Klein [13] or Dirac [6] eq. (5.1)) so that projective transformations are automatically transformation groups of the action (which of course allows to apply Klein’s Erlanger program in all details with respect to (linear) transformation groups and subgroups). We’ve had related $F^{\mu\nu}$ already to a (linear) Complex [4], so starting from the known rep $F^{\mu\nu}$, Ω in [6], eq. (5.1) suggests several interpretations and generalizations. Thus Ω can be interpreted in terms of incidence/conjugation of lines, involution of linear Complexes or generalized in terms of squares of two (general linear) Complexes respectively the most general possible form of a quadratic Complex. Moreover, this is not unique in that [6], eq. (5.1) allows to add other Complex reps as well as additional (incidence/conjugation) terms equating to 0 as long as the quadratic character of Ω is preserved¹⁰.

III. COMPLEXES AND SYMPLECTIC SYMMETRY

The notion ‘symplectic symmetry’ emerged because Weyl renamed the symmetry of the ‘Komplexgruppe’ to greek notion in order to avoid misconceptions and/or misunderstanding of complex

⁹ With respect to $SU(2) \times SU(2)$ structures, we still have to introduce the Lie mapping, see section IV.

¹⁰ In order to preserve the grade of a Complex under projective transformations, one has to use Plücker’s 5-dim (inhomogeneous) coordinates r, s, ρ, σ , and $\eta = r\sigma - s\rho$, or six homogeneous coordinates, *not* only the original 4-dim set of line coordinates r, s, ρ and σ ! This introduces the (quadratic) Plücker condition which may be used as an additional term while introducing an additional parameter into the action. Please note here for later use, that if we introduce a matrix notation $M := \begin{pmatrix} r & \rho \\ s & \sigma \end{pmatrix}$ then $\eta = \det M$. So at least for linear transformations of line coordinates this matrix rep provides a calculus (or symbolism) to transform line coordinates by performing matrix transformations.

numbers and Complexe¹¹. We use linear Complexe and their point rep as given e.g. by Hamermesh (see [8], ch. 10, eq. (10-73)).

A. (Linear) Symplectic Symmetry

If we do *not* split $n = 2\nu$ and the given decomposition into $(n - 2)$ -dim subspaces [8], but if instead we use general n and fix $n = 4$ in a linear enumeration of the coordinate reps x and y , we may write the bilinear form according to $x^T J y = x_\alpha^T J_{\alpha\beta} y_\beta$, $1 \leq \alpha, \beta \leq 4$, which (in this special case) leads to $x_1 y_2 - x_2 y_1 + x_3 y_4 - x_4 y_3$. Interpreting the coordinates as homogeneous coordinates, we may replace them by line coordinate reps (see e.g. [4], ch. 1.2), i.e. the invariance requirement of $x^T J y$ using point coordinates reads as the invariance of $p_{12} + p_{34}$ in terms of (homogeneous!) line coordinates $p_{\alpha\beta}$. Now, $p_{12} + p_{34}$ is a special case of a general linear Complex $A_{\alpha\beta} p_{\alpha\beta}$, and due to simplicity we'll use the expression $p_{12} \pm k p_{34} = 0$ for subsequent discussions, $k \in \mathbb{R}$ (for the moment) being nothing but a real parameter.

B. Special Relativity as Invariance Property

Having established the (symplectic) invariance requirement of $x^T J y = p_{12} + p_{34}$ and generalized it to $p_{12} \pm k p_{34} = 0$, we may ask for (point coordinate) transformations keeping such objects invariant. In selecting the two most trivial cases¹², it is obvious that at a first glance, we can keep $p_{\alpha\beta}$ invariant by itself when keeping the coordinates (and as such also the area element) $x_\alpha, y_\alpha \longrightarrow x'_\alpha = x_\alpha, y'_\alpha = y_\alpha$ invariant.

A second, almost trivial possibility is to apply transformations according to

$$\begin{aligned} x_1 &\longrightarrow x'_1 = x_1, & x_2 &\longrightarrow x'_2 = x_2, & y_1 &\longrightarrow y'_1 = y_1, & y_2 &\longrightarrow y'_2 = y_2, \\ x_3 &\longrightarrow x'_3 = x_3 \cosh \eta - x_4 \sinh \eta, & y_3 &\longrightarrow y'_3 = y_3 \cosh \eta - y_4 \sinh \eta, \\ x_4 &\longrightarrow x'_4 = x_4 \cosh \eta - x_3 \sinh \eta, & y_4 &\longrightarrow y'_4 = y_4 \cosh \eta - y_3 \sinh \eta. \end{aligned} \quad (2)$$

Direct calculation shows $(x'_3)^2 - (x'_4)^2 = x_3^2 - x_4^2$, or even $(x'_1)^2 + (x'_2)^2 + (x'_3)^2 - (x'_4)^2 = x_1^2 + x_2^2 + x_3^2 - x_4^2$, so the transformation (2) itself can be identified with a Lorentz transformation (see also [22], eq. (2.1.7)), with $\eta = \alpha$ denoting the rapidity and relating the 4- (or 0-) coordinate to the

¹¹ We hope to avoid such misery by strictly using capitalized 'C' and old german plural form in conjunction with the treatment of line Complexe.

¹² We can forget all background and think with respect to $p_{\alpha\beta}$ just in terms of 2×2 determinants of some point coordinates.

standard 'time' coordinate. So $\cosh \eta = \gamma$, $\sinh \eta = \gamma\beta$ and $\tanh \eta = \beta$ with $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$, $\beta = \frac{v}{c}$. The same holds, of course, for the second point y .

Now the interesting and important fact is that p_{34} (or $x_3y_4 - x_4y_3$) by itself is invariant¹³, too, i.e.

$$\begin{aligned} p'_{34} &= x'_3y'_4 - x'_4y'_3 \\ &\stackrel{(2)}{=} x_3y_4 (\cosh^2 \eta - \sinh^2 \eta) + x_4y_3 (\sinh^2 \eta - \cosh^2 \eta) \\ &= x_3y_4 - x_4y_3 \\ &= p_{34}. \end{aligned} \tag{3}$$

Without branching into the details of (Lorentz, Poincaré or general) group theory as usual at this point (see e.g. [22] or [9]), for us it is sufficient having derived this property from a (linear) Complex like $p_{12} \pm kp_{34} = 0$. Here, we do not have room to start the geometrical discussion of this Complex but we want to mention that this linear Complex incorporates already special choices (see e.g. [17], nrs. 28ff or [12], §§16, 17). Based mainly on Möbius's and Plücker's work (e.g. [16], [17], [18], [19] and references therein), Klein recalls point-plane mappings, null systems and line conjugation, and closes the gap to dynamics and second order surfaces (i.e. polar theory).

Defining the axis of the null system as conjugate (line) polar of an absolute line and using a special choice of an orthogonal coordinate system, Klein discussed in detail how to obtain the rep $p_{12} \pm kp_{34} = 0 \longrightarrow (xy' - x'y) + k(z - z') = 0$, k denoting the 'parameter of the null system' via $k = \frac{a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23}}{a_{12}^2 + a_{13}^2 + a_{14}^2}$. This notion is used as departure in various other contexts (see e.g. [1] or [24]) discussing aspects of (linear) line Complexes, sometimes using different notations. For our purpose, we use Lie's approach [14] departing from the interpretation of imaginary elements in projective geometry and using them to establish a mapping of two complexified planar real coordinates onto lines, i.e. a 4-dim planar space to 4-dim line space.

IV. LIE MAPPING (OR LIE TRANSFER)

While having the equivalent point/plane/line coordinate reps from within [4], section 1.4, in mind, we choose the line rep for further discussion. In the general case, we thus have to discuss the

¹³ In addition to direct calculation, one can use the 2×2 -determinant representation and standard rules of determinant calculus. This yields some more insight into standard rep theory if we 'transform' (or map) the determinant 'back' to the bilinear *point coordinate* 'invariant' $x^T y$ and use 'matrix' reps of group (or algebra) transformations, i.e. $y' = Gy$, $x'^T = (Gx)^T = x^T G^T$ with respect to the two columns. Also note that if we complexify the coordinates, i.e. $x_3, x_4, y_3, y_4 \in \mathbb{R} \longrightarrow \psi_1, \psi_2, \psi'_1, \psi'_2 \in \mathbb{C}$, we obtain expressions $\psi_1\psi'_2 - \psi_2\psi'_1$ to be compared with (spinorial) singlet structures using the same mapping onto matrices and e.g. $\psi = \psi'^+$ with hermitean conjugation. This, however, will be addressed in section IV in more detail.

(conjugation/reciprocity) relation of two linear Complexes (see e.g. [19], [10], [11], [20] II 18. Vortrag, and especially [14], §7ff) versus occurrences and use of quadratic Complexes. So Dirac's quest of finding a linear representation in p (or sloppy speaking 'the root' of p^2) can be cast onto various related and interwoven topics:

- Given a quadratic Complex \mathfrak{L}^2 , how can we determine an appropriate linear Complex \mathfrak{L} , its geometrical and its dynamical and transformation/covariance properties?
- Given a quadratic Complex \mathfrak{L}^2 , can we find other geometrical elements and/or representations \mathfrak{L} which square to a quadratic Complex, or what is the most general form of \mathfrak{L} , respectively?
- Determining the parameters of (planar) line pencils or the intersection points of lines e.g. with the absolute quadric, we obtain quadratic relations in terms of line coordinates. How can we relate them to physical observations and principles related to linear and quadratic line Complexes?

A. Complexification of 'Point' Coordinates

Based on Lie's work¹⁴ in line and Complex geometry [14], §4, number 11, one can consider general reciprocity mappings between two real 3-dim spaces r and R . For our purpose the specialization given in §7 eq. (1), based on choosing two line Complexes, is substantial. So points in r map to lines in R and vice versa. Following §§8 and 9, Lie obtained the special mapping¹⁵ $2\rho = X + iY, 2s = X - iY, 2\sigma = Z \pm H, 2r = -(Z \mp H) = -Z \pm H$ which constrains the linear Complex $r + \sigma$ in r mapped to minimal lines¹⁶ in R with $dX^2 + dY^2 + dZ^2 = 0$. Besides a lot of other features and results (see [14] or the overview in [12] §§25-27), this mapping accordingly transforms *lines* of the 3-dim real space r uniquely into *spheres* of R . Vice versa, a sphere of R described by $(X, Y, Z, H^2), \pm H = \sigma + r$, is transformed into only *two* lines of r ! Both lines

¹⁴ We use the notation 'Lie mapping' to denote the line-sphere mapping [14] which needs to be distinguished from 'Lie transformations' used in more recent phase or function space discussions by means of Lie series resp. power series expansions. If we want to emphasize the background notion towards higher geometry over the more technical mapping aspects, we use also the notation 'Lie transfer'. This shouldn't signal that especially reciprocity investigations or similar transfer mechanism haven't been performed before in geometrical considerations, as one can easily check in literature. However, Lie's special considerations provide a great contemporal overview and yield a beautiful application at the same time.

¹⁵ Standard small letters denote elements of the space r , capital letters those of R . Although r is used parallel with Plücker's line coordinate r , the respective context should suppress misunderstandings.

¹⁶ Which later have to be expressed in Pauli or spinorial counterparts.

$(X, Y, Z, +H)$ and $(X, Y, Z, -H)$ are polar with respect to the linear Complex $\pm H = \sigma + r = 0$ [14]. The uniqueness may be established by introducing the notion of orientation and Laguerre geometry, however, according to our earlier footnote by introducing the matrix M , we can represent M in terms of coordinates of R by

$$M = \begin{pmatrix} r & \rho \\ s & \sigma \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{2}(-Z \pm H) & \frac{1}{2}(X + iY) \\ \frac{1}{2}(X - iY) & \frac{1}{2}(Z \pm H) \end{pmatrix} = \frac{1}{2}(\pm H \mathbb{1} + X\sigma_1 - Y\sigma_2 - Z\sigma_3). \quad (4)$$

i.e. we obtain a mapping of lines (or line Complexes or even higher elements) of the space r into the space R . Lines of r are mapped onto Pauli matrices σ_α , $\sigma_0 = \mathbb{1}$, with real coefficients, or simply $SU(2)$. No need to recall the possibility to introduce quaternions here. However, it is apparent that higher line and Complex geometry have to find their appropriate counterparts in R space if lines are mapped to Pauli matrices (or quaternions). So a priori we expect appropriate 'reflections' of higher line and Complex geometry in Clifford algebras.

It is however noteworthy, by starting from Plücker's line equations $x = rz + \rho, y = sz + \sigma$ to mention one more fundamental issue where x and y denote (euclidean) *projections* of a line in space (parametrized by z) onto the xz and yz planes, respectively. $\eta \sim (ry - sz) = r\sigma - s\rho = \det M$ denotes the projection onto the xy -plane¹⁷. If we rewrite those equations in matrix rep, we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r & \rho \\ s & \sigma \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r & \rho \\ s & \sigma \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

after having introduced homogeneous (point) coordinates x_α . This requires to switch to quaternary matrix reps in order to describe transformations of the full set of homogeneous (point) coordinates x_α . It is obvious, that 2×2 inverse and/or conjugate matrices have to be considered when applying the Lie mapping, i.e. we expect 2×2 block structures in matrix reps of R transformations. Moreover, the 2×2 calculus of the Lie mapping lightens some background of expressions like $\vec{\sigma} \cdot \vec{v} \vec{\sigma} \cdot \vec{v} = \vec{v}^2$ on the context of line coordinates and Complex conjugation and/or quadratic Complexes.

At the time of writing, we tend to identify the space r with physical observable (projective) space whereas eq. (4) represents 'quantum' reps respectively R represents 'quantum space', e.g. in terms of 2×2 Pauli or 4×4 Dirac matrices considering two lines (or Complexes). However, it is important to note that this relation yields some assumptions related to linear (line) Complexes in

¹⁷ Please note once more the *alternative* rep of the line by $x(t)$, $y(t)$ and $z(t)$ or by a 3-dim euclidean 'vector' $\vec{x}(t)$ in terms of a time parameter and a related velocity 'vector', time-dependent or not. Both pictures can be 'completed' by appropriate (additional) homogeneous coordinates, however, it is necessary to care about the respective interpretations.

r [14], §7 and §8, and it is far from a general description especially if we relax the restrictions in r and proceed to general dynamics and higher geometry. Vice versa, we can establish a mapping to a Pauli/quaternionic or Clifford calculus in order to relate physical observations and observables.

In [14], Lie presents various further very interesting results, also with respect to Complex cones, differential geometry and mappings of tangent to curvature properties which we suppress here. However, there are two more important facts which Lie used to introduce contact interactions and his theory of partial differential equations but which for us is important to relate to physics. So he mentioned Klein's work on six Complexes in involution¹⁸ as a superset, in addition in [14], §9, number 28, he showed that two incident lines (being reciprocal/conjugate polars with respect to the special Complex $H = 0$) are mapped to two spheres being in contact, i.e. fulfilling $(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 = (H_1 \pm H_2)^2$. If we rewrite the (line) incidence relation like above (see also [4] section 2.1) and respect [19], [10], [11], the action can be treated by quadratic Complex theory. There is an interesting side effect when restricting observations to planar problems in that we may investigate (planar) Complex curves and still discuss (energy) conic sections in the planes as well as two pencils of lines related to two linear Complexes (and their congruence). We'll address this elsewhere.

B. Energy and Second Order Surfaces

Last not least it is noteworthy to focus on the description of physical actions which typically starts from an action principle and variations. From above, we are already equipped with an action principle; from the background of projective geometry, we are equipped with a coordinate tetrahedron, with transformation groups and (linear) representation theory, and last not least, we may use transfer principles.

So from above, we may include (invariant) products like $F^{\mu\nu}F_{\mu\nu}$ or $F^{a\mu\nu}F_{\mu\nu}^a$, $1 \leq a \leq 3$, and Complex geometry. However, we may as well try to find appropriate representations of line Complexes on other rep spaces. As such, we can observe that the product of two (real) quaternions a^+, b , $+$ denoting quaternionic conjugation, yields Complex coefficients if we neglect the trace for the moment. So the mapping $a^+ \cdot b \longrightarrow A_{\alpha\beta} q_\alpha^+ q_\beta \longrightarrow A_{\alpha\beta} q_\alpha \otimes q_\beta$ allows to extract the relevant (real) coefficients and relate them to $\text{SL}(2, \mathbb{H})$ and its real forms. On the other hand, placing two quaternions q^+, q into a 'spinor', we find of course a rotated 'spinor' set $1/2(q^+ \pm q)$ which

¹⁸ Recall the handedness of the null systems!

can be related to Dirac theory and Clifford algebra. In the background, it is of course the Lie mapping which is active in relating lines and Complexes to spheres and sphere Complexes, etc. For our purpose here, however, it is sufficient to justify the 'spinorial rep' of Complexes in an action principle which is based on the Lie mapping. For all calculations, the standard mechanisms can be applied to calculate energies and states, however, the interpretation has to be changed at a couple of places. We are going to investigate such topics in upcoming work.

References

- [1] Ball R S 1876. *The Theory of Screws: A Study in the Dynamics of a Rigid Body*. (Dublin: Hodges, Foster, and Co., Grafton-Street)
- [2] Dahm R 2010. *A Symmetry Reduction Scheme of the Dirac Algebra without Dimensional Defects*. *Yad. Fis.* **73** 297; *Phys. Atom. Nuclei* **73** 276
- [3] Dahm R 2012. *On A Microscopic Representation of Space-Time*. *Yad. Fis.* **75** 1244; *Phys. Atom. Nuclei* **75** 1173; <http://arxiv.org/abs/1102.0027>
- [4] Dahm R 2015. *On A Microscopic Representation of Space-Time III. ICGT 30, Ghent and ICCA 10, Tartu*, <http://arxiv.org/abs/1508.06872>, submitted to AACA
- [5] Dahm R 2015. *On A Microscopic Representation of Space-Time IV. QTS 9, Yerevan*, <http://arxiv.org/abs/1512.07119>, submitted to Yad. Fis./Phys. Atom. Nuclei
- [6] Dirac, P A M 1973. *Long range forces and broken symmetries*. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* **333** 403
- [7] Ehlers J, Pirani F and Schild A 1972. *The Geometry of Free Fall and Light Propagation*. In: *Studies in Relativity*. ed L O'Raifeartaigh (Oxford: Clarendon Press) pp 63–84
- [8] Hamermesh M 1962. *Group Theory*. (London-Paris: Pergamon Press)
- [9] Jackson J D 1983. *Klassische Elektrodynamik*. 2nd edition (1975, German translation) (Berlin: de Gruyter)
- [10] Klein F 1869. *Zur Theorie der Linien-Complexe des ersten und zweiten Grades*. *Math. Ann.* **2** 198; *Die allgemeine lineare Transformation der Linien-Co-ordinaten*. *Math. Ann.* **2** 366
- [11] Klein F 1871. *Notiz betreffend den Zusammenhang der Linien-Geometrie mit der Mechanik starrer Körper*. *Math. Ann.* **4** 403
- [12] Klein F 1926. *Vorlesungen über höhere Geometrie*. (Die Grundlehren der mathematischen Wissenschaft XXII) (Berlin: Springer)
- [13] Klein F 1928. *Vorlesungen über nicht-euklidische Geometrie*. (Die Grundlehren der mathematischen Wissenschaft XXVI) (Berlin: Springer)

- [14] Lie S 1872. *Ueber Complexe, insbesondere Linien- und Kugel-Complexe, mit Anwendung auf die Theorie partieller Differential-Gleichungen*. *Math. Ann.* **5** 145; *Ueber einige partielle Differential-Gleichungen zweiter Ordnung*. *Math. Ann.* **5** 209
- [15] Lie S 1896. *Geometrie der Berührungstransformationen*. (Leipzig: Teubner)
- [16] Möbius A F 1837. *Lehrbuch der Statik*. (Leipzig: Göschen)
- [17] Plücker J 1865. *On a new Geometry of Space*. *Phil. Trans. R. Soc. London* **155** 725
- [18] Plücker J 1866. *Fundamental Views regarding Mechanics*. *Phil. Trans. R. Soc. London* **156** 361
- [19] Plücker J 1868/1869. *Neue Geometrie des Raumes*. ed A Clebsch and F Klein (Leipzig: B. G. Teubner)
- [20] Reye Th 1899. *Die Geometrie der Lage I-III*. (4th edition, Leipzig: Baumgärtners Buchhandlung) (1866/1868)
- [21] Schoenflies A 1884. *Zur Theorie der Bewegung starrer räumlicher Systeme*. *Journal für die reine und angewandte Mathematik* **98** 265
- [22] Sexl R U, Urbantke H K 1992. *Relativität, Gruppen, Teilchen*. (3rd edition, Wien New York: Springer)
- [23] Study E 1903. *Geometrie der Dynamen*. (Leipzig: Teubner)
- [24] Zindler K 1902/1906. *Liniengeometrie mit Anwendungen*. (Sammlung Schubert XXXIV/LI) (Leipzig: G. J. Göschensche Verlagshandlung)